Chapter 1

Linearni diferencialni rovnice 1. radu

Postup reseni linearni diferencialni rovnice 1. radu $y' = a(x)y + b(x)$:

(i) Prevedeni rovnice na linearni diferencialni rovnici. Urceni funkci a a b.

(ii) Nalezeni funkci $A(x) = \int a(x)dx$ a $K(x) = \int b(x)e^{-A(x)}dx$.

(iii) Zapsani obecneho tvaru reseni $y(x) = e^{A(x)}(K(x) + K)$, $K \in \mathbb{R}$, $x \in I$, kde I je nejaky maximalni podinterval mnoziny $D_a \cap D_b$ (D_a je definicni obor funkce a).

(iv) V pripade, ze puvodni rovnice nebyla LDR, tak zjistit, zda se prevodem neztratily nejaka reseni, ci se nejaka reseni nedaji slepit.

(v) Pripadne vyreseni dodatecne podminky. Napriklad pocatecni podminky.

1.0.1 $y' - \frac{2y}{x} = 2x^3$

(i) $a(x) = \frac{2}{x}$, $b(x) = 2x^3$.

(ii) $A(x) = \int \frac{2}{x}dx = 2\log|x|$, $K(x) = \int 2xdx = x^2$.

(iii) Obecny tvar maximalniho reseni je $y_K^{\pm}(x) = (x^4 + Kx^2)$, $K \in \mathbb{R}$, $x \in \mathbb{R}^{\pm}$.

1.0.2 $y' = \frac{y}{x} - 1$

(i) $a(x) = \frac{1}{x}$, $b(x) = -1$.

(ii) $A(x) = \int \frac{1}{x}dx = \log|x|$, $K(x) = \int -\frac{1}{|x|}dx = -\operatorname{sign}(x)\log|x|$.

(iii) Obecny tvar maximalniho reseni je $y_K^{\pm}(x) = -x\log|x| + Kx$, $K \in \mathbb{R}$, $x \in \mathbb{R}^{\pm}$.

1.0.3 $y'x = y + x^2$

(i) $a(x) = \frac{1}{x}$, $b(x) = x$.

(ii) $A(x) = \int \frac{1}{x}dx = \log|x|$, $K(x) = \int \operatorname{sign}(x)dx = |x|$.

(iii) $y_K^{\pm}(x) = x^2 + Kx$, $K \in \mathbb{R}$, $x \in \mathbb{R}^{\pm}$.

(iv) V puvodni rovnici lze uvazovat $x = 0$ a pro tento bod lze reseni slepit. Obecny tvar maximalniho reseni tedy je $y_K(x) = x^2 + Kx$, $K \in \mathbb{R}$, $x \in \mathbb{R}$.

1.0.4 $y' + \frac{2y}{x} = \frac{e^{-x^2}}{x}$

(i) $a(x) = -\frac{2}{x}$, $b(x) = \frac{e^{-x^2}}{x}$.

(ii) $A(x) = \int -\frac{2}{x} dx = -2\log|x|$, $K(x) = \int xe^{-x^2} dx = -\frac{1}{2}e^{-x^2}$.

(iii) Obecny tvar maximalniho reseni tedy je $y_K^{\pm}(x) = \frac{K - \frac{1}{2}e^{-x^2}}{x^2}$, $K \in \mathbb{R}$, $x \in \mathbb{R}^{\pm}$.

1.0.5 $y'\cos(x) - y\sin(x) = \sin(2x)$

Tento priklad lze resit bud primo prevedenim na LDR a nebo substituci $z = y\cos(x)$. Budeme postupovat druhym zpusobem. Dostaneme tedy rovnici $z' = \sin(2x)$. Tedy $z(x) = -\frac{1}{2}\cos(2x) + K$, $K \in \mathbb{R}$, $x \in \mathbb{R}$. Pak $y_{K,k}(x) = \frac{z(x)}{\cos(x)} = -\frac{\cos(2x)+K}{2\cos(x)}$, $K \in \mathbb{R}$, $x \in \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right)$, $k \in \mathbb{Z}$. Pro $K = 1$ lze slepit a dostaneme maximalni reseni $y_1(x) = -\cos(x)$. Maximalni reseni pro $K \neq 1$ jsou jiz popsana vyse.

1.0.6 $xy' + y = \log(x) + 1$

Tento priklad lze resit bud primo prevedenim na LDR a nebo substituci $z = yx$. Budeme postupovat druhym zpusobem. Dostaneme tedy rovnici $z' = \log(x) + 1$. Tedy $z(x) = x\log(x) + K$, $K \in \mathbb{R}$, $x \in \mathbb{R}^+$. Pak maximalni reseni jsou $y_K(x) = \frac{z(x)}{x} = \log(x) + \frac{K}{x}$, $K \in \mathbb{R}$, $x \in \mathbb{R}^+$.

1.0.7 $y'(a^2 + x^2) + xy = 1$, $a \in \mathbb{R}$

(i) $a(x) = -\frac{x}{a^2+x^2}$, $b(x) = \frac{1}{a^2+x^2}$.

(ii) $A(x) = \int -\frac{x}{a^2+x^2} dx = -\frac{1}{2}\log(a^2 + x^2)$,

$$K(x) = \int \frac{1}{\sqrt{a^2 + x^2}} dx = \begin{cases} \operatorname{sign}(x)\log|x| : & a = 0 \\ \log\left(\sqrt{a^2 + x^2} + x\right) : & a \neq 0, \ ; \sqrt{a^2 + x^2} = t - x. \end{cases}$$

(iii) Maximalni reseni jsou $y_K^{0,\pm}(x) = \frac{\log(|x|)+K}{x}$, $K \in \mathbb{R}$, $x \in \mathbb{R}^{\pm}$ a $y_K^a(x) = \frac{\log\left(\sqrt{a^2+x^2}+x\right)+K}{x}$, $K \in \mathbb{R}$, $x \in \mathbb{R}$.

1.0.8 $(2x + 1)y' + y = x$

(i) $a(x) = -\frac{1}{2x+1}$, $b(x) = \frac{x}{2x+1}$. Oznacme $I_1 = \left(-\infty, -\frac{1}{2}\right)$ a $I_2 = \left(-\frac{1}{2}, +\infty\right)$.

(ii) $A(x) = \int -\frac{1}{2x+1} dx = -\frac{1}{2}\log|2x + 1|$,

$$K(x) = \int \frac{\operatorname{sign}(2x + 1)x}{\sqrt{|2x + 1|}} dx = \frac{1}{3}\sqrt{|2x + 1|}(x - 1), \text{ substituce } t = \sqrt{|2x + 1|}.$$

(iii) $y_K^i(x) = \frac{1}{3}(x - 1) + \frac{K}{\sqrt{|2x+1|}}$, $K \in \mathbb{R}$, $x \in I_i$.

(iv) V puvodni rovnici lze uvazovat $x = -\frac{1}{2}$ a pro tento bod lze reseni slepit pro $K = 0$. Obecny tvar maximalniho reseni pro $K \neq 0$ je popsan vyse. Maximalni reseni pro $K = 0$ je $y_0(x) = \frac{1}{3}(x - 1)$, $x \in \mathbb{R}$.

1.0.9 $y' - y\tan(x) = \mathrm{cotg}(x)$

(i) $a(x) = \tan(x)$, $b(x) = \mathrm{cotg}(x)$.

(ii) $A(x) = \int \tan(x)dx = -\log|\cos(x)|$,

$$K(x) = \mathrm{sign}(\cos(x)) \int \frac{1}{\sin(x)} - \sin(x)dx = \mathrm{sign}(\cos(x)) \left(\log\left|\tan\left(\frac{x}{2}\right)\right| + \cos(x)\right).$$

(iii) Maximalni reseni jsou $y_{K,k}(x) = 1 + \frac{\log\left|\tan\left(\frac{x}{2}\right)\right| + K}{\cos(x)}$, $K \in \mathbb{R}$, $x \in \left(k\frac{\pi}{2}, (k+1)\frac{\pi}{2}\right)$.

1.0.10 $y' + y\cos(x) = \sin(2x)$

(i) $a(x) = -\cos(x)$, $b(x) = \sin(2x)$.

(ii) $A(x) = \int -\cos(x)dx = -\sin(x)$,

$$K(x) = \int e^{\sin(x)}\sin(2x)dx = 2(\sin(x) - 1)e^{\sin(x)}, \text{ substituce } t = \sin(x).$$

(iii) Maximalni reseni jsou $y_K(x) = 2(\sin(x) - 1) + Ke^{-\sin(x)}$, $K \in \mathbb{R}$, $x \in \mathbb{R}$.

1.0.11 $y' + \frac{x+1}{x}y = 3xe^{-x}$

(i) $a(x) = -\frac{x+1}{x}$, $b(x) = 3xe^{-x}$.

(ii) $A(x) = \int -\frac{x+1}{x}dx = -x - \log|x|$, $K(x) = \int 3|x|xdx = |x^3|$.

(iii) Maximalni reseni jsou $y_K^{\pm}(x) = (x^2 + \frac{K}{x})e^{-x}$, $K \in \mathbb{R}$, $x \in \mathbb{R}^{\pm}$.